Quantum Anti–de Sitter Space at Roots of Unity

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Abstract

An algebra of functions on q–deformed Anti-de Sitter space AdS_q^D with star-structure is defined for roots of unity, which is covariant under $U_q(so(2, D-1))$. The scalar fields have an intrinsic high–energy cutoff, and arise most naturally on products of the quantum AdS space with a classical sphere. Hilbert spaces of scalar fields are constructed.

1   Introduction

The aim of these notes is to give a short summary of the paper [6]. We define and study a non–commutative version of the D–dimensional Anti–de Sitter (AdS) space which is covariant under the standard Drinfeld–Jimbo quantum group $U_q(so(2, D-1))$, for q a root of unity and $D \geq 2$. The symmetry group $SO(2, D-1)$ plays the role of the D–dimensional Poincaré group. Part of the motivation for doing this is an interesting conjecture relating string or M theory on $AdS^D \times W$ with (super)conformal field theories on the boundary [4], where W is a certain sphere or a product space containing a sphere. There is in fact some evidence that a full quantum treatment would lead to some non–classical version of the manifolds. This includes the appearance of a "stringy exclusion principle" [7] in the spectrum of fields on AdS space. In view of this and the well–known connections of conformal field theory with quantum groups at roots of unity, it seems plausible that some kind of quantum AdS space at roots of unity should be relevant to string theory in the above context. We argue in [6] that the space presented below follows quite uniquely form general covariance assumptions.

It is well–known that at roots of unity, quantum groups show completely new, "non–perturbative" features. In particular, there exist finite–dimensional unitary representations of the quantum AdS groups at roots of unity [5, 2], which consistently combine all the features of the classical one–particle representations with a high–energy cutoff. In particular, scalar fields which are unitary representations of $U_q(so(2, D-1))$ will be obtained as polynomials in the coordinate functions. Moreover, it turns out that the quantum spaces are obtained most naturally as products $AdS_q^D \times S^D/\Gamma$ if D is odd, and $AdS_q^D \times S_\chi^{2D-1}/\Gamma$ if D is even.

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2   The structure of AdS_q^D

The algebra of functions on quantum AdS space is a real sector of the complex orthogonal quantum sphere [1]

$$(P^-)_{kl}^{ij} t_i t_j = 0, \qquad g^{ij} t_i t_j = R^2 \tag{2.1}$$

where P^- is the q–antisymmetrizer [1]. This is covariant under $\mathcal{U}_q^{res} := U_q^{res}(so(2, D-1))$, which has generators $\{H_i, X_i^\pm\}$ which satisfy the standard Drinfeld–Jimbo commutation relations, plus additional generators of a *classical* Lie algebra which is either $so(D+1)$ if D is odd, or $sp(D)$ if D is even; compare [5]. The reality condition to specify the appropriate noncompact form is $H_i^* = H_i$, $X_i^{\pm *} = (-1)^E X_i^\mp (-1)^E = s_i X_i^\mp$ with $s_i = \pm 1$, where E is the generator of $U_q(so(2, D-1))$ corresponding to the energy. We consider roots of unity $q = e^{i\pi/M}$, and define $M_S = M/d_S$ where $d_S = 2$ if D is even, and $d_S = 1$ if D is odd.

Consider the space of irreducible polynomials of degree n in the generators, $\mathcal{F}(n) = \mathcal{U}_q^{res} \cdot (t_1)^n$. This is an irreducible highest–weight representation of $\mathcal{U}_q^{res}$, and it is unitary with respect to the compact form $U_q^{res}(so(D+1))$ if

$$n < k_S := M_S - (D-2)/2. \tag{2.2}$$

Hence in that case, $\mathcal{F}(n)$ should be considered as a space of functions on the compact quantum sphere S_q^{D-1}. However if $M_S \le n < M_S + k_S$, then $\mathcal{U}_q^{fin} \cdot (t_1)^n$ is an irreducible unitary representation of $\mathcal{U}_q^{fin} = U_q^{fin}(so(2, D-1))$, and can be identified with a square–integrable scalar field with the same lowest weight (i.e. mass) on the classical AdS space. Here $\mathcal{U}_q^{fin}$ is the subalgebra of $\mathcal{U}_q^{res}$ with generators $\{H_i, X_i^\pm\}$, which becomes the usual AdS group in the classical limit. Therefore it make sense to define the Hilbert space of (positive–energy, square–integrable) functions on quantum AdS space by

$$AdS_q^D := \bigoplus_{M_S \le n < M_S + k_S} \mathcal{U}_q^{fin} \cdot (t_1)^n. \tag{2.3}$$

From a physical point of view, this describes spin 0 particles.

The coordinate functions t_i define operators $\hat{t}_i$ on this Hilbert space, which induces a star structure on them by the operator adjoint. It can be written explicitly in the form $\hat{t}_i^* = -\Omega(-1)^E \hat{t}_j C_i^j \Omega(-1)^E$, where Ω is a unitary element with $\Omega^2 = 1$, which implements the the longest element of the Weyl group.

Upon closer examination, it turns out that the space of ("almost") all polynomial functions in the generators t_i decomposes as

$$\oplus_n \mathcal{F}(n) = \left(AdS_q^D \times M/\Gamma \right) \oplus \left(S_q^D \times M/\Gamma \right) \oplus \ldots, \tag{2.4}$$

where $M = S^D$ or S_χ^{2D-1}, and M/Γ denotes the space of functions on some (twisted) orbifold; see [6] for more details. This means that the space which arises most naturally is a product

of quantum AdS space with an orbifold of a classical sphere or of a "chiral" sphere, plus other sectors. The symmetries of these additional spheres are implemented by the classical generators which are contained in the full algebra $U_q^{res}(so(2, D-1))$, as mentioned above.

One can also look at this additional structure from an other point of view. For example in the case $D = 4$, it turns out that the smallest irreducible representation of $U_q^{res}(so(2,3))$ which contains a scalar field as in (2.3) on quantum AdS space is in fact a 4–dimensional multiplet of $sp(4)$, which is in a sense spontaneously broken to $su(2) \times su(2)$. In other words, there are in general additional "global" symmetries, and the sum of all these multiplets in $\oplus \mathcal{F}(n)$ becomes a sector of a classical sphere.

Furthermore, we give an argument in [6] that this quantum AdS space has an intrinsic length scale of order $L_{min} = R/M_S$, where the geometry is expected to become non–classical. One can also define a differential calculus and integration. A formulation of field theory on this space should hence be possible along the lines of [3].

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